Mathematical Modeling - Fall 2020, Final Project

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David Shih

1 Tacoma Narrows Bridge

 # Hooke's Law # Structure Engineering
A mathematical model that attempts to capture the Tacoma Narrows Bridge incident was proposed by McKenna and Tuama [2001]. The goal is to explain how torsional, or twisting, oscillations can be magnified by forcing that is strictly vertical. Consider a roadway of width 2ℓ hanging between two suspended cables, as in following Figure 6.18(a). We will consider a two-dimensional slice of the bridge, ignoring the dimension of the bridge's length for this model, since we are only interested in the side-to-side motion. At rest, the roadway hangs at a certain equilibrium height due to gravity; let y denote the current distance the center of the roadway hangs below this equilibrium. Hooke's law postulates a linear response, meaning that the restoring force the cables apply will be proportional to the deviation. Let θ be the angle the roadway makes with the horizontal. There are two suspension cables, stretched $y - \ell \sin\theta$ and $y + \ell \sin\theta$ from equilibrium, respectively. Assume that a viscous damping term is given that is proportional to the velocity. Using Newton's law $F = ma$ and denoting Hooke's constant by K, the equations of motion for y and θ are as follows:

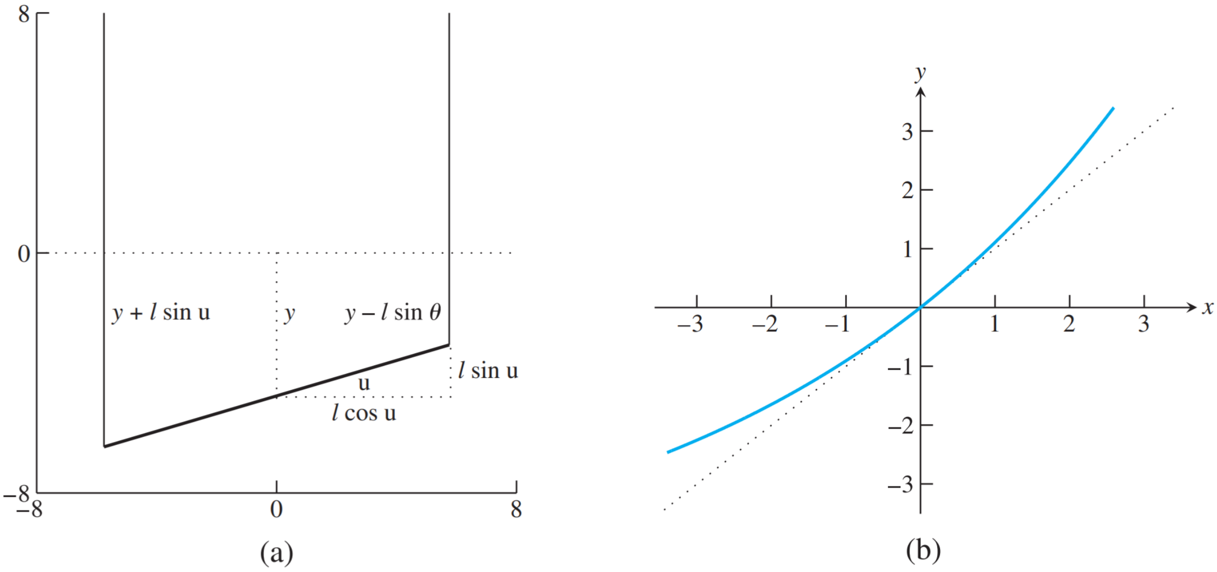


Figure 6.18 Schematics for the McKenna–Tuama model of the Tacoma Narrows Bridge. (a) Denote the distance from the roadway center of mass to its equilibrium position by y, and the angle of the roadway with the horizontal by θ. (b) Exponential Hooke's law curve $f(y) = (K/a)(e^{ay} - 1)$.

$$y'' = -dy' - \left[\frac{K}{m}(y - \ell \sin\theta) + \frac{K}{m}(y + \ell \sin\theta)\right],$$

$$\theta'' = -d\theta' + \frac{3\cos\theta}{\ell}\left[\frac{K}{m}(y - \ell \sin\theta) + \frac{K}{m}(y + \ell \sin\theta)\right].$$

As the equations stand, the state $y = y' = \theta = \theta' = 0$ is an equilibrium. Now turn on the wind. Add the forcing term $0.2W \sin\omega t$ to the right-hand side of the y equation, where W is the wind speed in km/hr. This adds a strictly vertical oscillation to the bridge.
How to build the model and is the model good enough to show the truth? If the model are not good enough, how to improve it?

2 Heat Distribution of Cooling Fin

 # Poisson Equation # Thermal science

Heat sinks are used to move excess heat away from the point where it is generated. In this project, the steady-state distribution along a rectangular fin of a heat sink will be modelled. The heat energy will enter the fin along part of one side. The main goal will be to design the dimensions of the fin to keep the temperature within safe tolerances.

The fin shape is a thin rectangular slab, with dimensions $L_x \times L_y$ and width δ cm, where δ is relatively small. Due to the thinness of the slab, we will denote the temperature by $u(x, y)$ and consider it constant along the width dimension.

Heat moves in the following three ways: conduction, convection, and radiation. Conduction refers to the passing of energy between neighboring molecules, perhaps due to the movement of electrons, while in convection the molecules themselves move. Radiation, the movement of energy through photons, will not be considered here.

Conduction proceeds through a conducting material according to Fourier's first law

$$q = -KA\nabla u,$$

where q is heat energy per unit time (measured in watts), A is the cross-sectional area of the material, and ∇u is the gradient of the temperature. The constant K is called the thermal conductivity of the material. Convection is ruled by Newton's law of cooling,

$$q = -HA(u - u_b),$$

where H is a proportionality constant called the convective heat transfer coefficient and u_b is the ambient temperature, or bulk temperature, of the surrounding fluid (in this case, air).

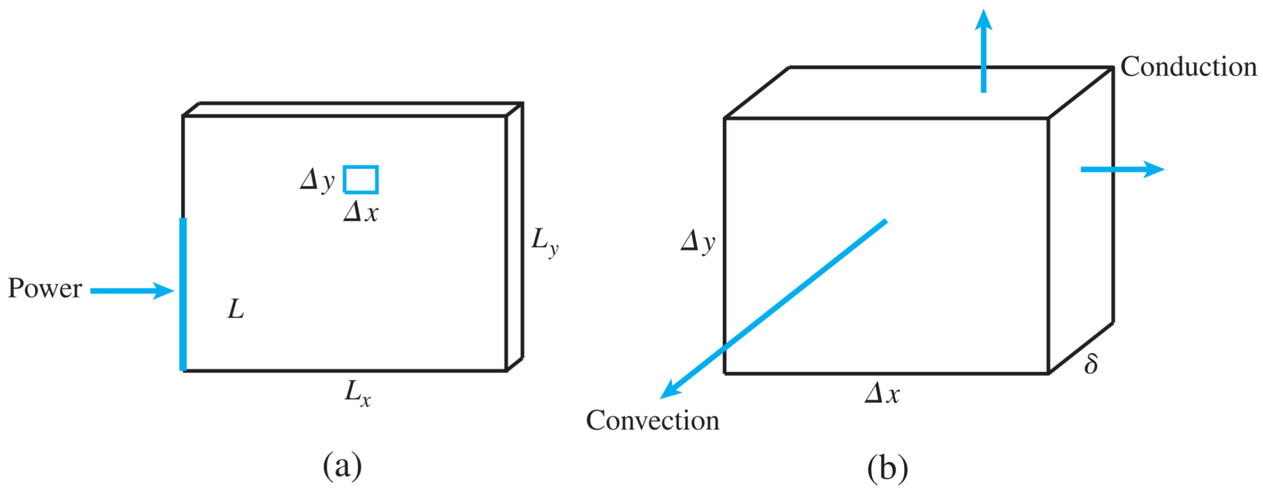


Figure 8.15 Cooling fin in Reality Check 8. (a) Power input occurs along interval $[0, L]$ on left side of fin. (b) Energy transfer in small interior box is by conduction along the x and y directions, and by convection along the air interface.

Assume that the fin is composed of aluminium, whose thermal conductivity is $K = 1.68\text{W}/\text{cm}^\circ\text{C}$ (watts per centimeter-degree Celsius). Assume that the convective heat transfer coefficient is $H = 0.005\text{W}/\text{cm}^{2\circ}\text{C}$, and that the room temperature is $u_b = 20^\circ\text{C}$.

Cooling fins for desktop, we have many method to improve cooling, for some example

- With different material, copper or aluminium (or multiple).
- With water cooling system.
- With different shape of heat sink.
- With fans to improve convection of air.

How to build the model for those types of cooling system?

3 Heat Transfer with Material Changing Phase

 # Thermal science
Changing the phase when temperature increasing (like boiling, melting, sublimation) will change a lot of things. For example, thermal conductivity and CTE (coefficient of thermal expansion), will have different trend for each material. And both of it can cause lots of different phenomena in the heat transfer question.
Consider a heat transfer system on one dimension, with Dirichlet boundary condition on both side. Temperature in the heating process will cross the point that change the phase (melting point or boiling point). Build an mathematical model for this kind of phenomena to show the detail of each timing.

4 Population of Nonconformist (Protestantism)

 # Social Sciences # Religion
In a book entitled *Looking at History Through Mathematics*, Rashevsky [Ra], pp. 103–110, considers a model for a problem involving the production of nonconformists in society. Suppose that a society has a population of $x(t)$ individuals at time t, in years, and that all nonconformists who mate with other nonconformists have offspring who are also nonconformists, while a fixed proportion r of all other offspring are also nonconformist. If the birth and death rates for all individuals are assumed to be the constants b and d, respectively, and if conformists and nonconformists mate at random, the problem can be expressed by the differential equations

$$\frac{dx(t)}{dt} = (b-d)x(t) \quad \text{and} \quad \frac{dx_n(t)}{dt} = (b-d)x_n(t) + rb(x(t) - x_n(t))$$

where $x_n(t)$ denotes the number of nonconformists in the population at time t.

a. Suppose the variable $p(t) = x_n(t)/x(t)$ is introduced to represent the proportion of nonconformists in the society at time t. Show that these equations can be combined and simplified to the single differential equation

$$\frac{dp(t)}{dt} = rb(1 - p(t)).$$

b. Assuming that $p(0) = 0.01$, $b = 0.02$, $d = 0.015$, and $r = 0.1$, approximate the solution $p(t)$ from $t = 0$ to $t = 50$ when the step size is $h = 1$ year.

c. Solve the differential equation for $p(t)$ exactly, and compare your result in part (b) when $t = 50$ with the exact value at that time.

By the previous experiment, how to build the model for the population of each religion in Taiwan?

5 Cattle and Grass (and others?)

 # Biological interaction # Animal Migration
They are many type of different biological interaction, like predation, mutualism, commensalism, parasitism, competition. Most of the time, we talk about the predator and prey model (Lotka–Volterra equations). But, in real world, the population of each species are not just only impact by this relation. For example, cattle and grass are in the different relation, this kind of relation are dominated by the location are fertile or not or where the predators are.
Try to build a model for cattle and grass, and base on that model, give some more real world phenomena to make the model more complicated.

Hint: Let $D \subseteq \mathbb{R}^2$ be the region of interest, and $u(x, y, t)$, $v(x, y, t)$ denote the number of cattle and the amount of grass at location (x, y) and time t. What kind of differential equation can u and v satisfy?

6 Moving Inclined Plane

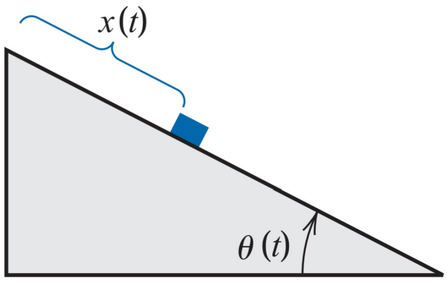
 # Centrifugal Force # Friction
A particle starts at rest on a smooth inclined plane whose angle θ is changing at a constant rate

$$\frac{d\theta}{dt} = \omega < 0$$

At the end of t seconds on the end of a plane (length L), we have some relation by free diagram

$$\frac{d^2x(t)}{dt^2} = -g\sin(\omega t) - \omega^2(L - x(t)).$$

Suppose the particle has moved 0.5182 m in 1 s. Find, to within 10^{-5}, the rate ω at which θ changes. Assume that $g = 9.8$ m/s^2.



In previous cases we have constant of angular velocity ω, now we want to know if the angular velocity changing smoothly (function $\omega(t)$), and add the force of friction. Give a model to show the trajectory of the given particle. And try to use the model you build to find out following question.

- If we set particle at different location on the plane, how far can we throw that particle out? And, how to set up the function $\omega(t)$?
- Can we set a function $\omega(t)$ to make the particle stay on the plane, after a full rotation?